SUBSPACES AND QUOTIENT SPACES OF $(\Sigma G_n)_{l_p}$ AND $(\Sigma G_n)_{c_0}$

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ABSTRACT

Let (G_n) be a sequence which is dense (in the sense of the Banach-Mazur distance coefficient) in the class of all finite dimensional Banach spaces. Set $C_p = (\Sigma G_n)_{l_p} (1 < p < \infty)$, $C_\infty = (\Sigma G_n)_{c_0}$. It is shown that a Banach space X is isomorphic to a subspace of C_p $(1 < p \leqq \infty)$ if and only if X is isomorphic to a quotient space of C_p.

1. Introduction

In [3] it was shown that if X is a subspace of a quotient space of C_p $(1 < p \leqq \infty)$ and X has a shrinking, finite dimensional decomposition (FDD, in short), then X is isomorphic to a space of the form $(\Sigma E_n)_{l_p}$ $(1 < p < \infty)$ or $(\Sigma E_n)_{c_0}$ $(p = \infty)$ with dim $E_n < \infty$. It follows easily from this that if $X$is a subspace of a quotient space of C_p and X^* has the approximation property, then X is isomorphic to a complemented subspace of C_p. In particular, such an X is isomorphically both a quotient space and a subspace of C_p. Since it is clear that if X is a complemented subspace of C_p $(1 < p \leqq \infty)$ then X and X^* have the approximation property, a somewhat different approach seemed to be required for studying subspaces and quotient spaces of C_p which fail the approximation property. (Of course, Davie's construction [1] gives that C_p has subspaces and quotient spaces which fail the approximation property.)

However, it turns out that the results of [3] can be used to study arbitrary quotient spaces (and, by duality, subspaces) of C_p, if one utilizes the technique of [2, Th. IV.4]. This theorem asserts that if X^* is separable, then X has a subspace

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Y so that Y and X/Y each have a shrinking, finite dimensional decomposition. Thus if X is a quotient space of C_p, then Y and X/Y each have an l_p (c_0, if $p = \infty$) FDD by the results of [3]. Then one uses a variation of an observation of Lindenstrauss to show that X is a subspace of C_p.

Our notation and terminology agree with [3].

2. The main results

The observation of J. Lindenstrauss mentioned in the introduction is the following:

Let Y be a subspace of a Banach space X and assume that Y and X/Y embed into c_0. Then X embeds into c_0.

Indeed, let $T: Y \to c_0$ and $S: X/Y \to c_0$ be isomorphic embeddings into c_0 and let $Q: X \to X/Y$ be the quotient map. Then, since c_0 has the separable extension property (refer to [5]) and X is separable, there is an operator $\tilde{T}: X \to c_0$ with $\tilde{T}|_Y = T$. One easily checks that the operator $U: X \to c_0 \oplus c_0$ defined by $Ux = (\tilde{T}x, SQx)$ is an isomorphism into.

THEOREM 1. *Let X be a quotient space of C_p $(1 < p \leqq \infty)$. Then Xs i isomorphic to a subspace of C_p.*

PROOF. Let us first treat the case $p = \infty$. If X is a quotient of C_∞ then X^* is separable, so by [2, Th. IV.4] there is a subspace Y of X so that both Y and X/Y have shrinking FDDs. But then, by [3, Prop. 1], Y and X/Y both have c_0 decompositions, hence they both embed into c_0 and therefore, by Lindenstrauss' observation above, X embeds into c_0.

Assume now that $1 < p < \infty$. The argument here is similar to the argument when $p = \infty$, but we must repeat some arguments used in the proof of [2, Th. IV.4] to avoid the use of the separable extension property.

First, let (x_n, x_n^*) be a shrinking Markuschevich basis for X (that is, $x_n^*(x_k) = \delta_{n,k}$, $[x_n] = X$ and $[x_n^*] = X^*$; Mackey [4] proved that such a system exists in every space whose dual is separable). Next, choose integers $1 = m_1 < m_2 < m_3 < \cdots$ so that

(1) if $x \in [x_i]_{i=1}^{m_n}$ and $y \in [x_i]_{i=m_{n+1}-1}^{\infty}$ then $\| x \| \leqq (1 + n^{-1}) \| x + y \|$

and

(2) if $x^* \in [x_i^*]_{i=1}^{m_n}$ and $y^* \in [x_i^*]_{i=m_{n+1}-1}^{\infty}$ then $\| x^* \| \leqq (1 + n^{-1}) \| x^* + y^* \|$.

To see that this is possible, assume that $1 = m_1 < m_2 < \cdots < m_n$ have been defined. Since $[x_n] = X$ and $[x_n^*] = X^*$ we may pick m_{n+1} large enough so that for

$x \in [x_i]_{i=1}^{m_n}$, $\| x \| \leqq (1 + n^{-1}) \sup\{x^*(x): x^* \in [x_i^*]_{i=1}^{m_{n+1}-1}, \| x^* \| = 1\}$ and for $x^* \in [x_i^*]_{i=1}^{m_n}$,

$$\| x^* \| \leqq (1 + n^{-1}) \sup\{x^*(x): x \in [x_i]_{i=1}^{m_{n+1}-1}, \| x \| = 1\}.$$

It is easy to see that $\{m_i\}_{i=1}^{n+1}$ satisfy (1) and (2).

Now set $E_n = [x_i]_{i=m_n}^{m_{n+1}-1}$ and $F_n = [x_i^*]_{i=m_n}^{m_{n+1}-1}$. Clearly (1) implies that (E_{p_n}) is an FDD whenever $p_n + 1 < p_{n+1}$, so [3, Prop. 1] gives that there is an increasing sequence (k_n) of positive integers so that (E_{k_n}) is an l_p decomposition. Set

$$Z_n = X/[E_1 + E_2 + \cdots + E_{k_n-2} + E_{k_n+2} + E_{k_n+3} + \cdots]$$

and let $Q_n: X \to Z_n$ be the quotient map. Note that

$$Z_n^* = [E_1 + \cdots + E_{k_n-2} + E_{k_n+2} + \cdots]^{\perp} = F_{k_n-1} + F_{k_n} + F_{k_n+1},$$

so we may assume (since we can pass to a subsequence of (k_n)), in view of (2) and [3, Prop. 1] that $(F_{k_n-1} + F_{k_n} + F_{k_n+1})$ is an l_q decomposition ($p^{-1} + q^{-1} = 1$). Define now $Q: X \to (\Sigma Z_n)_{l_p}$ by $Qx = (Q_n x)$. Since $(F_{k_n-1} + F_{k_n} + F_{k_n+1})$ is an l_q decomposition it follows easily that Q^* is an isomorphism of $(\Sigma Z_n^*)_{l_q}$ onto $[F_{k_n-1} + F_{k_n} + F_{k_n+1}]$. Thus Q is continuous (and, incidentally, also onto). Note that the restriction of Q to $[E_{k_n}]$ is an isomorphism because (E_{k_n}) is an l_p decomposition, $QE_{k_n} \subset Z_n$, (Z_n) is an l_p decomposition and, by (1), $Q|_{E_{k_n}}$ is an isomorphic embedding whose inverse has norm $\leqq 6$. Finally, set $Y = [E_{k_n}]$. We claim that $Y^{\perp} = [F_i]_{i \notin (k_n)}$. Obviously $[F_i]_{i \notin (k_n)} \subset Y^{\perp}$; to prove equality, let

$$y_j^* = \sum_{i=m_j}^{m_{j+1}-1} y^*(x_i)x_i^*.$$

Then $y_{k_n}^* = 0$ for all n and, by (2), the sequence

$$\left(\sum_{n=1}^{N} \sum_{j=k_n+1}^{k_{n+1}-1} y_j^*\right)_{N=1}^{\infty}$$

is bounded. Clearly, this sequence converges w^* to y^* and because the sequence

$$\left(\sum_{j=k_n+1}^{k_{n+1}-1} y_j^*\right)_{n=1}^{\infty}$$

forms a basis for its span, by (2), the reflexivity of X implies that

$$\sum_{n=1}^{N} \sum_{j=k_n+1}^{k_{n+1}-1} y_j^*$$

converges to y^* in norm, by R.C. James' well-known theorem on boundedly complete bases. It follows that $y^* \in [F_i]_{i \notin (k_n)}$. Letting $G_n = [F_i]_{i=k_n+1}^{k_{n+1}-1}$, we have that (G_n) is an FDD for $Y^\perp$. Hence by [3, Prop. 1], $Y^\perp$ has an l_q FDD, whence (since $(X/Y)^* = Y^\perp$) X/Y has an l_p FDD. Following the idea in Lindenstrauss' observation above, we define $S: X \to (\Sigma Z_n)_{l_p} \oplus X/Y$ by $Sx = (Qx, x + Y)$. Then, since $Q|_Y$ is an isomorphism, S is an isomorphism. Since $(\Sigma Z_n)_{l_p} \oplus X/Y$ embeds into C_p, this completes the proof of Theorem 1.

REMARK. It is clear that C_∞ embeds into c_0, so every quotient space of c_0 is isomorphic to a subspace of c_0. The corresponding statement for l_p is false; however, if (H_n) is a sequence of spaces which is dense (in the sense of the Banach-Mazur distance) in the set of all finite dimensional quotient spaces of subspaces of l_p, then the proof of Theorem 1 shows that every quotient space of l_p embeds into $(\Sigma H_n)_{l_p}$.

We prove now the converse to Theorem 1.

THEOREM 2. *Let X be a subspace of C_p $(1 < p \leqq \infty)$. Then X is isomorphic to a quotient space of C_p.*

PROOF. For $1 < p < \infty$ the assertion follows easily from Theorem 1 by duality. Indeed, if X is a subspace of C_p, $1 < p < \infty$, then X^* is a quotient space of C_q $(p^{-1} + q^{-1} = 1)$ hence by Theorem 1, X^* is isomorphic to a subspace of C_q and therefore $X = X^{**}$ is isomorphic to a quotient space of C_p.

Assume now that X is a subspace of C_∞. We repeat the argument of Theorem 1 to show that there is an isomorphism V from X^* into C_1. Since X is non-reflexive, we must exercise some care to make sure that V is w^* continuous, so that X will be isomorphic to a quotient of C_∞.

Now since X^* is separable, there is a shrinking Markuschevich basis (x_n, x_n^*) in X and, as in the proof of Theorem 1, we select integers $1 = m_1 < m_2 < m_3 < \cdots$ so that (1) and (2) are satisfied, and we define E_n and F_n in exactly the same way as before. We will show that if $1 = k_1 < k_2 < \cdots$ is a sequence of integers thin enough then the following conditions are satisfied:

(3) $$[F_{k_n}]_{n=1}^\infty = [E_i]_{i \notin (k_n)}^\perp$$

(4) $$(E_{k_n-1} + E_{k_n} + E_{k_n+1})_{n=1}^\infty \text{ is a } c_0 - \text{FDD}$$

and

(5) $$(F_{k_n})_{n=1}^\infty \text{ is an } l_1 - \text{FDD}.$$

As soon as (3)–(5) are satisfied, the proof is along the same lines as the previous one. Indeed, by (1) and the fact that $[x_n^*] = X^*$, we have that

$$([E_i]_{i=k_n+1}^{k_{n+1}-1})_{n=1}^{\infty}$$

is a shrinking FDD, hence by [3, Prop. 1], there is a blocking (U_n) of

$$([E_i]_{i=k_n+1}^{k_{n+1}-1})_{n=1}$$

such that (U_n) is a c_0 decomposition. Let $Q_n\colon U_n \to X$ be the natural isometric embedding and define $Q\colon (\Sigma U_n)_{c_0} \to X$ by $Q(\Sigma u_n) = \Sigma Q_n u_n$, where $u_n \in U_n$. Obviously Q is an isomorphic embedding and hence $Q^*\colon X^* \to (\Sigma U_n^*)_{l_1}$ is onto. The kernel of Q^* is $[E_i]_{i \notin k_n}^{\perp}$ which is equal to $[F_{k_n}]_1^{\infty}$ by (3). Let

$$Z_n = X^*/[F_i]_{i \neq k_n-1, k_n\, k_n+1};$$

then $Z_n^* = [E_{k_n-1} + E_{k_n} + E_{k_n+1}]$ for each n. Set $T_n\colon Z_n^* \to X$ to be the natural isometric embedding and let $T\colon (\Sigma Z_n^*)_{c_0} \to X$ be defined by $T(\Sigma z_n^*) = \Sigma T_n z_n^*$ $(z_n^* \in Z_n^*)$. Then, in view of (4), T is an isomorphic embedding and hence T^* is a map of X^* onto $(\Sigma Z_n)_{l_1}$. Moreover, since $T^*|_{F_{k_n}}$ is an isomorphism whose inverse has norm $\leqq 6$ we obtain that $T^*|_{[F_{k_n}]_1^{\infty}}$ is an isomorphism because of (5). Finally, define $V\colon X^* \to (\Sigma U_n^*)_{l_1} \oplus (\Sigma Z_n)_{l_1}$ by $Vx^* = (Q^*x^*, T^*x^*)$. It is easy to verify that V is an isomorphic embedding (note that if $\| x^* \| = 1$ and $\mathrm{dist}(x^*, [F_{k_n}]_1^{\infty})$ is big then $\| Q^*x^* \|$ is big since the kernel of Q^* is $[F_{k_n}]_1^{\infty}$; while if $\mathrm{dist}(x^*, [F_{k_n}]_1^{\infty})$ is small then $\| T^*x^* \|$ is big because $T^*|_{[F_{k_n}]_1^{\infty}}$ is an isomorphism).

Since V is w^* continuous there is an onto map $S\colon (\Sigma U_n)_{c_0} \oplus (\Sigma Z_n)c_0 \to X$ satisfying $S^* = V$, hence X is a quotient of C_{∞}.

It remains to show that if (k_n) is thin enough, then (3)–(5) are satisfied. We start with (4). Given any sequence q_n with $q_n + 2 < q_{n+1} - 1$, it follows from (1) that $(E_{q_n-1} + E_{q_n} + E_{q_n+1})$ is an FDD which, since $[x_n^*] = X^*$, must be shrinking. Thus by [3, Prop. 1], there is a subsequence (k_n) of (q_n) (with $k_1 = q_1$) so that $(E_{k_n-1} + E_{k_n} + E_{k_n+1})$ is a c_0 FDD. This proves (4).

To prove (3) we use again the technique of [2, p. 90]. Let $k_n + 2 < k_{n+1}$ and let $S_n\colon X^* \to [F_{k_i}]_{i=1}^n$ be the natural projection defined by

$$S_n x^* = \sum_{j=1}^{n} \sum_{i=p_j}^{p_{j+1}-1} x^*(x_i)x_i^*$$

where $p_j = m_{k_j}$. Obviously S_n is linear, bounded, and w^* continuous. Also, for every $x^* \in [E_i]_{i \notin (k_n)}^{\perp}$, $\| S_n x^* \| \leqq (1 + n^{-1}) \| x \|^*$. Indeed, if n is fixed we have from (2) that

$$\| S_{n|F} \| \leqq 1 + n^{-1}, \text{ where } F = [(F_{k_j}: 1 \leqq j \leqq n) \cup (F_i)_{i=k_{n+1}}^{\infty}].$$

But it is clear that F is contained in $(E_i: i < k_{n+1}$ and $i \notin (k_j))^{\perp}$. Since both these spaces have the same finite codimension, they are equal. Thus $\| S_n x^* \| \leqq (1 + n^{-1}) \| x^* \|$ for $x^* \in [E_i]_{i \notin (k_n)}^{\perp}$. It is proved in [2, p. 90] that for every

$$x^* \in [E_i]_{i \notin (k_n)}^{\perp} \; S_n x^* \xrightarrow{w^*} x^*.$$

In addition, in view of the above inequality, $\| S_n x^* \| \to \| x^* \|$. Since $X \subseteq C_{\infty}$, X has the Kadec-Klee property (that is, whenever

$$y^*, y_n^* \in X^*, \; y_n \xrightarrow{w^*} y^*$$

and $\| y_n^* \| \to \| y \|$, then also $\| y_n^* - y^* \| \to 0$) and therefore $\| S_n x^* - x^* \| \to 0$ for all $x^* \in [E_i]_{i \notin (k_n)}^{\perp}$. We thus have that $(F_{k_n})_1^{\infty}$ is an FDD for $[E_i]_{i \notin (k_n)}^{\perp}$ and (3) is proved.

Note that

$$(X / [E_i]_{i \notin (k_n)})^* = [E_i]_{i \notin (k_n)}^{\perp} = [F_{k_n}] \text{ and } X / [E_i]_{i \notin (k_n)}$$

is a quotient of a subspace of C_{∞}. Hence the argument of [2, p. 91] and our [3, Prop 1] imply that $X / [E_i]_{i \notin (k_n)}$ has a c_0-FDD (G_m) determined by projections (P_m) such that the dual l_1-FDD $(P_m^*[F_{k_n}]_1^{\infty})_{m=1}^{\infty}$ consists of disjoint blocks $[F_{k_i}]_{i=q_m}^{q_{m+1}-1}$ of (F_{k_n}). It is now clear that if for every integer m we pick $q_m \leqq i_m < q_{m+1}$ then the thinner subsequence, $(F_{k_{i_m}})$ will be an l_1-FDD. This proves (5) and completes the proof of Theorem 2.

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